APPROXIMATE ANGLE FACTORS FOR TWO-DIMENSIONAL PROBLEMS

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UDC 536.12

We have approximated the special $Ki_n(x)$ functions used in formulas for radiation characteristics. We have found the approximate intermediate functions for an isotropic incident flow by means of which [2] many angle factors have been expressed.

§ 1. Introduction

Two-dimensional systems of bodies with an absorbing medium exhibit comparitively simple radiation characteristics. For the isotropic radiation of surfaces, Mikk has examined the angle factors in a number of papers [1-3]. Many of the factors are determined by means of the intermediate functions M, N_1, N_2, and S_2, and these in turn are expressed in terms of the tabulated Bessel functions and their integrals. The radiation characteristics have been generalized in [4] for the axisymmetric indicatrix of effective surface radiation, given by a series in cosines. The formulas have been simplified by using the special $Ki_n(x)$ functions in the place of the Bessel functions. Moreover, the above-cited reference enumerates most fully the properties of the Ki_n functions. The difficulties are now reduced to the calculations of the Ki_n functions. Tables of these functions are not readily accessible and they are limited.

To use an electronic digital computer, particularly machines of the Promin and Nairi types, the functions must be approximated by simple formulas. Here we offer approximate Ki_n functions whose derivatives are intermediate functions, and we give important examples of the approximation of angle factors.

§ 2. Approximate $Ki_n(x)$ Functions

The original formulas are taken in two variants:

$$\mathrm{Ki}_n(x) = \int_0^1 \frac{\mu^{n-1}}{\sqrt{1-\mu^2}} \exp\left(-\frac{x}{\mu}\right) d\mu,$$

$$\mathrm{Ki}_n(x) = \int_0^1 (1-\xi^2)^{\frac{n-2}{2}} \exp\left(-\frac{x}{\sqrt{1-\xi^2}}\right) d\xi. \tag{1}$$

The integrals are replaced by quadrature formulas

$$\mathrm{Ki}_n(x) \simeq \sum_m a_i \exp(-b_i x). \tag{2}$$

As the first approximation the coefficients a_i and b_i are determined from the weights and nodes of the Gauss quadrature. As a rule, the coefficients a_i are then increased and simultaneously rounded off to satisfy the equation

$$\sum_m a_i = \mathrm{Ki}_n(0) = \sqrt{\pi}\,\Gamma(n/2)/2\Gamma[(n+1)/2].$$

Urals Electromechanical Institute for Railroad Transportation Engineers, Sverdlovsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 3, pp. 499-503, March, 1969. Original article submitted May 3, 1968.

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TABLE 1. Coefficients of the Binomial Functions $Ki_n(x)$ and Their Maximum Errors in the Interval $0 \le x \le 3$

n	a_i	b_i	$\delta\,\%\,(x)$
3	0,65 0,1354	1,08 2,12	0,34 (1)
4	0,08667 0,58	1,9 1,068	0,13 (1) —0,08 (3)
5	0,089 0,5	1,59 1,048	0,12 (1,5)

TABLE 2. Coefficients of the Trinomial Functions $Ki_n(x)$ and Their Maximum Errors in the Interval $0 \le x \le 3$

n	a_i	b_i	$\delta\,\%\,(x)$
1	0,28 0,6 0,6908	8,9 1,88 1,06	—0,8 (1)
2	0,05 0,32 0,63	6 1,75 1,06	—0,02 (0,1) 0,04 (0,3) —0,15 (0,7) 0,24 (2)
3	0,0354 0,25 0,5	3,5 1,42 1,04	0,05 (0,3) —0,26 (1,5) 0,16 (3)
4	0,0167 0,17 0,48	3 1,4 1,039	0,02 (0,1) —0,04 (0,3) —0,04 (2)
5	0,00505 0,124 0,46	3 1,4 1,038	—0,06 (0,5) 0,09 (3)
6	0,2708 0,2499 0,0126	1,007 1,155 2,168	—0,16 (1,5)

Then, by comparison with exact tables compiled for the interval $0 \le x \le 3$ [5] we correct the coefficients b_i, making use of the fact that they can be rounded off. We chose between formula (2) and the original formula (1), selecting the one which exhibits the smaller error. Tables 1 and 2 show the coefficients of (2) and the maximum errors in the interval $0 \le x \le 3$. The trinomial formulas for n = 3-5 have been derived from quadrinomial formulas, compiled on the basis of the same quadrature. One of the four terms is dropped, with the remaining coefficients adjusted to fit. This method achieves minimum error, and its effectiveness increases in proportion to the increase in n. For larger orders of n the trinomial formulas should be derived from pentanomial formulas. The smaller the terms that are dropped, the easier the adjustment.

§ 3. Approximation of the Intermediate Functions

The intermediate functions for the calculation of the angle factors in the case of isotropic radiation of a surface have the form

$$M(x) = \frac{4}{\pi} \mathrm{Ki}_3(x),$$

$$N_1(x) = \frac{4}{\pi} [\mathrm{Ki}_1(x) - \mathrm{Ki}_3(x)],$$

$$N_2(x) = \frac{4}{\pi} [\mathrm{Ki}_2(x) - \mathrm{Ki}_4(x)],$$

$$S_1(x) \equiv 2E_3(x) = \int_0^1 \exp(-xt)\, t^{-3} dt,$$

$$S_2(x) = \frac{4}{\pi} \int_0^1 \mathrm{Ki}_3(xt) \frac{t\,dt}{\sqrt{1-t^2}}.$$

All of these functions are also presented in the form

$$\sum_m a_i \exp(-b_i x).$$

For M, N_1, and N_2 the coefficients a_i and b_i are determined, in first approximation, from the coefficients of the functions $Ki_n(x)$. The adjustment was subsequently carried out by comparing the results against the tabulated material [3]. In a number of instances the tables were quite detailed. The coefficients $E_3(x)$ in first approximation are taken from the Gauss quadrature. The calculations for the function S_2 proved to be most difficult. For this function, the first-approximation coefficients were also taken from the Gauss quadrature with m = 4. The integrand Ki_3 has been replaced by a binomial formula. The resulting sum of eight terms has been reduced by half. The results are given in Table 3. Here we also find the interval with the error being investigated. Beyond the limits of this interval, for the functions M and N_2 the last significant figure is confirmed in the table [3]. The values of the quadrinomial function S_2 for $x > 6$ are markedly underestimated.

For arguments $x \le 1, 2$ it is best to present the Ki_n functions in the form of a series. It was demonstrated earlier [4] that the error in this case amounts to tenths of a percent. On the basis of the Ki_n functions, for $0 \le x \le 1, 3$, we have obtained

$$N_2(x) = \frac{4}{3\pi} - x + \frac{1}{3}x^3 + \frac{4}{\pi}[0.30793x^2 - 0.055141x^4 - 0.001454x^6 - x^2 \ln x\,(0.5 - 0.022x^2)],$$

$$S_2(x) = 1 - x + \frac{2}{3}x^2 - \left(0.25758 - \frac{\ln x}{8}\right)x^3 - (5.1 - 2.5\ln x)\,10^{-3}x^5. \qquad (3)$$

Each of these formulas gives a deviation of less than one in the fourth significant figure.

TABLE 3. Coefficients of the Formulas Approximating the Intermediate Functions

Function	a_i	b_i	Max. error δ % (x)	Test interval
$M(x)$	0,046	3,4	−0,22 (1,5)	
	0,317	1,42	0,66 (5)	$0 \leqslant x \leqslant 5$
	0,637	1,04		
$N_2(x)$	0,0144	15	−0,3 (0,3)	$0 \leqslant x \leqslant 1,5$
	0,127	3,07	0,17 (1)	
	0,21	1,5	−0,53 (1,5)	
	0,073	1,077		
$N_1(x)$	0,36	8,5	−1,2 (0,1)	
	0,53	1,9	2 (0,3)	—
	0,11	1,08	−1,3 (0,7)	
$S_2(x)$	0,2	2	0,05 (0,3)	
	0,8	0,75	−0,22 (1)	$0 \leqslant x \leqslant 2$
			1,9 (2)	
$S_2(x)$	0,035	0,206	−0,22 (0,2)	
	0,235	0,51	0,44 (1,3)	$0 \leqslant x \leqslant 5$
	0,56	1,04	−1,36 (4,5)	
	0,17	1,57	2,3 (6)	

§ 4. The Most Important Examples of Calculating Angle Factors

a) The functions $S_2(x)$ and $2E_3(x)$ have the sense of local and average angle factors for a circular cylinder and infinite parallel surfaces. $M(x)$ is the local hemispherical angle factor for a circular cylinder and a point at the center.

b) The average angle factor for the parallel faces of the beam is calculated [2] from the approximation formula

$$\varphi_{AB} = \frac{\varphi_0}{\sqrt{1+h^2}} [\varphi_0 2E_3(uh) + hM(uh)],$$

where $\varphi_0 \equiv \varphi_{AB}$ when $u = 0$; $\varphi_0 = \sqrt{1 + h^2} - h$; $u = k\Delta$; $h = H/\Delta$; Δ is the width of the sides; H is the distance between the sides, k is the attenuation factor. Since the formula yields an underestimated result, we have used a slightly overstated function $E_3(x)$ in the interval [0, 2], and namely, we assumed that $b_1 = 8$, $b_3 = 1.123$ (in the place of 8.6 and 1.125). The results are compared with the table for φ_{AB}, compiled from the exact formula in [6]. For the region of variation in the arguments $0.05 \leq u \leq 2$, $0.2 \leq h \leq 10$ the formula yields understated results. The maximum error is 2.6% for $u = 1$, $h = 0.5$. For most numbers it amounts to tenths of a percent. The error beyond the field of the table as $u \to 0$ and $h \to 0$ must approach zero. This is remarkable because in the integral for which the table was compiled [6], as $h \to 0$, there is a singularity which markedly reduces the reliability of the result. The singularity does not yet show up in the field of the table. We have noted a typographical error in the compilation of the table. For $h = 3.5$ and $u = 1.5$, $\varphi_{AB} = 0.000271$ instead of 0.0005.

c) For the average angle factor for the perpendicular sides of the bar Mikk found the exact formula

$$\varphi_{AC} = \frac{1}{2u} \left[\frac{4}{3\pi} + N_2\left(u\sqrt{1+h^2}\right) - N_2(u) - N_2(uh) \right].$$

Calculations on the basis of this formula, using our approximate functions N_2, were compared with the table of [6], derived by numerical integration. If the function $N_2(x)$ is taken in the form $\sum\limits_{4} a_i \exp(-b_i x)$ with the coefficients from Table 3, over the entire field of the table from [6] ($0.005 \leq u \leq 2$, $0.2 \leq h \leq 10$ the error does not exceed 1.67% ($u = 0.05$, $h = 0.8$). In the overwhelming majority of cases (roughly, when $uh > 0.15$) it is less than 0.5%. The use of formulas for $N_2(x)$ from (3) yields good results, but in a limited interval of argument (roughly, when $uh < 1$-1, 2). Even here, the integral for which the table in [6] has been compiled exhibits a singularity as $h \to 0$, whereas both of the approximate formulas yield a guaranteed result.

d) The local and average angle factors for the two coaxial cylinders are calculated [2] from the approximate formula

$$\varphi_{rR} = \frac{1}{\rho}\{(\rho-1)M[(\rho-1)r] + 2E_3[(\rho-1)r]\},$$

where $\rho = R/r$; r and R are the minor and major radii, multiplied by the attentuation factor. Data with respect to the exact formula of [7] have been published for $1.05 \le \rho \le 3$ and $0.1 \le r \le 5$. Comparison shows that on the whole, as ρ and r are increased, the error increases from hundredths to tenths of a percent. Roughly, the inequality $r\rho \le 4$ limits the region with an error of $\le 2\%$.

In conclusion, we note that in 1) all of the calculations have been performed on the Promin computer, which functions with five digits and a floating decimal point (in a number of cases, there was noticeable round-off error); 2) the cited method of approximate calculation can be used for numerous other coefficients, including the coefficients for surfaces with nonisotropic axisymmetric radiation; 3) the error δ_0 in the approximate formulas of Mikk, as a rule, was considerably lower than 3%. But is has been determined for the transmission capacity. We have everywhere indicated the error in δ for the values of the functions. For angle factors $\delta = \delta_0\varphi_0/\varphi$, where φ_0 is the coefficient for k = 0. When δ_0 = 1-2% the quantity δ may amount to tenths of a percent.

NOTATION

$Ki_n(x)$	denotes the special functions (1);
M, N_1, N_2, and S_2	are intermediate functions, introduced in [1, 2];
$E_3(x)$	is the integral exponential function of third order;
a_i and b_i	are the coefficients of the approximation polynomial;
Γ	is the gamma function;
φ and φ_0	are the angle factors for a system with an absorbing and a transparent medium;
$u = k\Delta$;	
$h = H/\Delta$;	
Δ and H	are the width and the height of the beam;
k	is the attenuation factor, m^{-1};
r and R	are the radii, multiplied by the attenuation factor;
$\rho = R/r$.	

Subscripts and Superscripts

AB denotes the coefficients for the parallel faces of the beam;
AC denotes the coefficients for the perpendicular faces of the beam.

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